Formalising Contemporary Mathematics in Simple Type Theory

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"No matter how much wishful thinking we do, the theory of types is here to stay. There is *no other way* to make sense of the foundations of mathematics. Russell (with the help of Ramsey) had the right idea, and Curry and Quine are very lucky that their unmotivated formalistic systems are not inconsistent."

–Dana Scott (1969)

From simple type theory to proof assistants for higher-order logic

- Russell (1910), Ramsey (1926), etc.
- Church's typed λ-calculus (1940) formalisation
- *base types* including the Booleans, and function types
- sets and specifications (e.g. $\mathbb{N}$) coded as *predicates* (and sometimes as types)
- Wenzel (1997): axiomatic type classes

Advantages over dependent types

- Simpler syntax, semantics, and therefore *implementations*
- … which therefore can give us **more automation**
- Fewer **surprises** with hidden arguments, type checking
- HOL is *self-contained*: inductive definitions, recursion, etc are reducible to the base logic
- **Extensional equality** for sets and functions

But is formalised maths possible?

Whitehead and Russell needed 362 pages to prove 1+1=2!

We have better formal systems than theirs.

Gödel proved that all reasonable formal systems must be incomplete!

But mathematicians also work from axioms!

Church proved that first-order logic is undecidable!

We want to **assist** people, not to **replace** them.

Mathematics in Isabelle/HOL

- Jordan curve theorem
- Central limit theorem
- Residue theorem
- Prime number theorem
- Gödel's incompleteness theorems
- Algebraic closure of a field
- Verification of the Kepler conjecture*

- Matrix theory, e.g. Perron–Frobenius
- Analytic number theory, eg Hermite–Lindemann
- Nonstandard analysis
- Homology theory
- Topology
- Complex roots via Sturm sequences
- Measure, integration and probability theory

Distinctive features of Isabelle/HOL

- Simple types with axiomatic type classes
- Powerful automation: proofs *and* counterexamples
- Structured proof language
- Interactive development environment (PIDE)
- User-definable mathematical notation
- "Literate" proof documents can be generated in LaTeX
- An archive of over ***600 proof developments; 385 authors*** and nearly ***3 million lines of code***

Can we do *set theory* in higher-order logic?

- HOL is actually weaker than **Zermelo** set theory
- … but we can simply add a type of ZF sets with the usual axioms.
- [our framework presupposes the *axiom of choice*]
- … and develop cardinals, ordinal arithmetic, order types and the rest.

Partition notation: $\alpha \longrightarrow (\beta, \gamma)^n$

$[A]^n$ denotes the set of unordered n-element sets of elements of A

if $[\alpha]^n$ is partitioned ("coloured") into two parts (0, 1) then there's either

- a subset $B \subseteq \alpha$ of order type β whose n-sets are all coloured by 0
- a subset $C \subseteq \alpha$ of order type γ whose n-sets are all coloured by 1

Infinite Ramsey theorem: $\omega \longrightarrow (\omega, \omega)^n$

Erdős' problem (for 2-element sets)

$\alpha \longrightarrow (\alpha,2)$ is trivial

$\alpha \longrightarrow (|\alpha|+1, \omega)$ **fails** for $\alpha > \omega$

So which countable ordinals α satisfy $\alpha \longrightarrow (\alpha,3)$?

It turns out that α must be a power of ω

In 1987, Erdős offered a $1000 prize for a full solution

Material formalised for this project

$\omega^2 \longrightarrow (\omega^2, m)$ (Specker)

$\omega^{1+\alpha n} \longrightarrow (\omega^{1+\alpha}, 2^n)$ (Erdős and Milner)

$\omega^\omega \longrightarrow (\omega^\omega, m)$ (Milner, Larson)

Plus background theories: Cantor normal form for ordinals; facts about order types; the Nash-Williams partition theorem

Project done with Mirna Džamonja and Angeliki Koutsoukou-Argyraki

Paul Erdős and E. C. Milner, 1972

$\omega^{1+\alpha n} \longrightarrow (\omega^{1+\alpha}, 2^n)$ for α an ordinal and n a natural number

"We have known this result since 1959"
(it's in Milner's 1962 PhD thesis)

It's a five-page paper that needed a **full-page correction** in 1974.

We now conclude the proof of the theorem.

Since β is denumerable and nonzero, there is a sequence $(\gamma_n : n<\omega)$ which repeats each element of B infinitely often, i.e. such that

(11) $$|\{n : \gamma_n = \nu\}| = \aleph_0 \qquad (\nu \in B).$$

Since tp $S=\alpha\beta$, we may write $S=S^{(0)}=\bigcup\,(\nu \in B)A_\nu^{(0)}(<)$.

Let $n<\omega$ and suppose we have already chosen elements $x_i \in S(i<n)$ and a subset

(12) $$S^{(n)} = \bigcup(\nu \in B)A_\nu^{(n)}(<)$$

of S of order type $\alpha\beta$. Since α is right-SI, $A_{\gamma_n}^{(n)}$ contains a final section A' such that $A_{\gamma_n}^{(n)} \cap \{x_0, \ldots, x_{n-1}\}<A'$. By (10), there are $x_n \in A'$, a strictly increasing map $g_n : B \to B$ and sets $A_\nu^{(n+1)}(\nu \in B)$ such that

(13) $$g_n(\gamma_i) = \gamma_i \qquad (i \leq n),$$

(14) $$A_\nu^{(n+1)} \subset K_1(x_n) \cap A_{g_n(\nu)}^{(n)} \qquad (\nu \in B).$$

From the definition of A', it follows that

(15) $$x_n \in A_{\gamma_n}^{(n)} \subset S^{(n)}$$

and

(16) $$x_i < x_n \quad \text{if} \quad i < n \quad \text{and} \quad x_i \in A_{\gamma_n}^{(n)}.$$

$S^{(n+1)}$ is defined by equation (12) with n replaced by $n+1$. It follows by induction that there are x_n, $A_\nu^{(n)}(\nu \in B)$, $S^{(n)}$ and g_n such that (12)–(16) hold for $n<\omega$.

Let $Z=\{x_n : n<\omega\}$. If $m<n<\omega$, then by (15), (14), and (12) we have that

Key steps of Erdős and Milner's proof

- Every ordinal is a "strong type" *(about 200 lines of machine proof)*
- A "remark" about indecomposable ordinals *(72 lines)*
- A key lemma: $\alpha\beta \longrightarrow (\min(\gamma, \omega\beta), 2k)$ if $\alpha \longrightarrow (\gamma, k)$ for $k \geq 2$ *(about 960 lines)*
- The main theorem $\omega^{1+\alpha n} \longrightarrow (\omega^{1+\alpha}, 2^n)$ by induction on n *(about 30 lines)*
- Larson's corollary: $\omega^{nk} \longrightarrow (\omega^n, k)$ *(about 35 lines)*

Every ordinal is a “strong type”

We will say that β is a strong type if, whenever tp $B = \beta$ and $D \subset B$, then there are $n < \omega$ and sets $D_1, \ldots, D_n \subset D$ such that

(5) tp D_i *is*

(6) *if* $M \subset$

```
proposition strong_ordertype_eq:
  assumes D: "D ⊆ elts β" and "Ord β"
  obtains L where "⋃(List.set L) = D" "⋀X. X ∈ List.set L ⟹ indecomposable (tp X)"
    and "⋀M. ⟦M ⊆ D; ⋀X. X ∈ List.set L ⟹ tp (M ∩ X) ≥ tp X⟧ ⟹ tp M = tp D"
proof -
  define φ where "φ ≡ inv_into D (ordermap D VWF)"
  then have bij_φ: "bij_betw φ (elts (tp D)) D"
    using D bij_betw_inv_into down ordermap_bij by blast
  have φ_cancel_left: "⋀d. d ∈ D ⟹ φ (ordermap D VWF d) = d"
    by (metis D φ_def bij_betw_inv_into_left down_raw ordermap_bij small_iff_range total_on_
  have φ_cancel_right: "⋀γ. γ ∈ elts (tp D) ⟹ ordermap D VWF (φ γ) = γ"
    by (metis φ_def f_inv_into_f ordermap_surj subsetD)
  have "small D" "D ⊆ ON"
    using assms down elts_subset_ON [of β] by auto
  then have φ_less_iff: "⋀γ δ. ⟦γ ∈ elts (tp D); δ ∈ elts (tp D)⟧ ⟹ φ γ < φ δ ⟷ γ < δ"
    using ordermap_mono_less [of _ _ VWF D] bij_betw_apply [OF bij_φ] VWF_iff_Ord_less φ_can
    by (metis ON_imp_Ord Ord_linear2 less_V_def order.asym)
```

A remark about indecomposable ordinals

```
proposition indecomposable_imp_Ex_less_sets:
  assumes indec: "indecomposable α" and "α > 1" and A: "tp A = α" "small A" "A ⊆ ON"
    and "x ∈ A" and A1: "tp A1 = α" "A1 ⊆ A"
  obtains A2 where "tp A2 = α" "A2 ⊆ A1" "{x} ≪ A2"
proof -
  have "Ord α"
    using indec indecomposable_imp_Ord by blast
  have "Limit α"
    by (simp add: assms indecomposable_imp_Limit)
  define φ where "φ ≡ inv_into A (ordermap A VWF)"
  then have bij_φ: "bij_betw φ (elts α) A"
    using A bij_betw_inv_into down ordermap_bij by blast
  have bij_om: "bij_betw (ordermap A VWF) A (elts α)"
    using A down ordermap_bij by blast
  define γ where "γ ≡ ordermap A VWF x"
  have γ: "γ ∈ elts α"
    unfolding γ_def using A ‹x ∈ A› down by auto
  then have "Ord γ"
    using Ord_in_Ord ‹Ord α› by blast
  define B where "B ≡ φ ` (elts (succ γ))"
  show thesis
  proof
    have "small A1"
      by (meson ‹small A› A1 smaller_than_small)
    then have "tp (A1 - B) ≤ tp A1"
```

If $x \in A$ and $A_1 \subseteq A$, with type $A, A_1 = \alpha$, then there is $A_2 \subseteq A_1$ such that $\{x\} < A_2$.

$\alpha\beta \longrightarrow (\min(\gamma, \omega\beta), 2k)$ if $\alpha \longrightarrow (\gamma, k)$

- Assume there is no $X \in [\alpha\beta]^{2k}$ such that $[X]^2$ is 1-coloured
- Assume there is no $C \subseteq \alpha\beta$ of order type γ such that $[C]^2$ is 0-coloured
- Then show there is a $Z \subseteq \alpha\beta$ of order type $\omega\beta$ such that $[Z]^2$ is 0-coloured

this will require generating an ω-chain of sets of type β

```
theorem Erdos_Milner_aux:
  assumes part: "partn_lst_VWF α [ord_of_nat k, γ] 2"
    and indec: "indecomposable α" and "k > 1" "Ord γ" and β: "β ∈ elts ω1"
  shows "partn_lst_VWF (α*β) [ord_of_nat (2*k), min γ (ω*β)] 2"
proof (cases "α=1 ∨ β=0")
  case True
  show ?thesis
  proof (cases "β=0")
    case True
    moreover have "min γ 0 = 0"
      by (simp add: min_def)
    ultimately show ?thesis
      by (simp add: partn_lst_triv0 [where i=1])
  next
    case False
    then obtain "α=1" "Ord β"
      by (meson ON_imp_Ord Ord_ω1 True β elts_subset_ON)
    then obtain i where "i < Suc (Suc 0)" "[ord_of_nat k, γ] ! i ≤ α"
      using partn_lst_VWF_nontriv [OF part] by auto
    then have "γ ≤ 1"
      using ‹α=1› ‹k > 1› by (fastforce simp: less_Suc_eq)
    then have "min γ (ω*β) ≤ 1"
      by (metis Ord_1 Ord_ω Ord_linear_le Ord_mult ‹Ord β› min_def order_trans)
    moreover have "elts β ≠ {}"
      using False by auto
    ultimately show ?thesis
      by (auto simp: True ‹Ord β› ‹β≠0› ‹α=1› intro!: partn_lst_triv1 [where i=1])
  qed
next
  case False
  then have "α ≠ 1" "β ≠ 0"
    by auto
```

$$\alpha\beta \longrightarrow (\min(\gamma, \omega\beta), 2k)$$

$$\text{if } \alpha \longrightarrow (\gamma, k)$$

Equation (8) with its one-line proof

(8) *If A ⊂ S, then there is X ∈ [*

This follows from the hypothesi

```
have Ak0: "∃X ∈ [A]⇗k⇖. f ` [X]⇗2⇖ ⊆ {0}" —‹remark (8) about @{term"A ⊆ S"}›
  if A_αβ: "A ⊆ elts (α*β)" and ot: "tp A ≥ α" for A
proof -
  let ?g = "inv_into A (ordermap A VWF)"
  have "small A"
    using down that by auto
  then have inj_g: "inj_on ?g (elts α)"
    by (meson inj_on_inv_into less_eq_V_def ordermap_surj ot subset_trans)
  have Aless: "⋀x y. ⟦x ∈ A; y ∈ A; x < y⟧ ⟹ (x,y) ∈ VWF"
    by (meson Ord_in_Ord VWF_iff_Ord_less ‹Ord(α*β)› subsetD that(1))
  then have om_A_less: "⋀x y. ⟦x ∈ A; y ∈ A; x < y⟧ ⟹ ordermap A VWF x < ordermap A VWF y
    by (auto simp: ‹small A› ordermap_mono_less)
  have α_sub: "elts α ⊆ ordermap A VWF ` A"
    by (metis ‹small A› elts_of_set less_eq_V_def ordertype_def ot replacement)
  have g: "?g ∈ elts α → elts (α * β)"
    by (meson A_αβ Pi_I' α_sub inv_into_into subset_eq)
  then have fg: "f ∘ (λX. ?g ` X) ∈ [elts α]⇗2⇖ → {..<2}"
    by (rule nsets_compose_image_funcset [OF f _ inj_g])
  have g_less: "?g x < ?g y" if "x < y" "x ∈ elts α" "y ∈ elts α" for x y
    using Pi_mem [OF g]
    by (meson A_αβ Ord_in_Ord Ord_not_le ord ‹small A› dual_order.trans elts_subset_ON inv_
  obtain i H where "i < 2" "H ⊆ elts α"
    and ot_eq: "tp H = [k,γ]!i" "(f ∘ (λX. ?g ` X)) ` (nsets H 2) ⊆ {i}"
    using ii partn_lst_E [OF part fg] by (auto simp: eval_nat_numeral)
  then consider (0) "i=0" | (1) "i=1"
    by linarith
  then show ?thesis
  proof cases
    case 0
    then have "f ` [inv_into A (ordermap A VWF) ` H]⇗2⇖ ⊆ {0}"
      using ot_eq ‹H ⊆ elts α› α_sub by (auto simp: nsets_def [of _ k] inj_on_inv_into elim
    moreover have "finite H ∧ card H = k"
```

Main Theorem

```
theorem Erdos_Milner:
  assumes ν: "ν ∈ elts ω1"
  shows "partn_lst_VWF (ω↑(1 + ν * ord_of_nat n)) [ord_of_nat (2^n), ω↑(1+ν)] 2"
proof (induction n)
  case 0
  then show ?case
    using partn_lst_VWF_degenerate [of 1 2] by simp
next
  case (Suc n)
  have "Ord ν"
    using Ord_ω1 Ord_in_Ord assms by blast
  have "1+ν ≤ ν+1"
    by (simp add: ‹Ord ν› one_V_def plus_Ord_le)
  then have [simp]: "min (ω ↑ (1 + ν)) (ω * ω ↑ ν) = ω ↑ (1+ν)"
    by (simp add: ‹Ord ν› oexp_add min_def)
  have ind: "indecomposable (ω ↑ (1 + ν * ord_of_nat n))"
    by (simp add: ‹Ord ν› indecomposable_ω_power)
  show ?case
  proof (cases "n = 0")
    case True
    then show ?thesis
      using partn_lst_VWF_ω_2
  next
    case False
    then have "Suc 0 < 2 ^ n"
      using less_2_cases not_less_eq by fastforce
    then have "partn_lst_VWF (ω ↑ (1 + ν * n) * ω ↑ ν) [ord_of_nat (2 * 2 ^ n), ω ↑ (1 + ν)] 2"
      using Erdos_Milner_aux [OF Suc ind, where β = "ω↑ν"] ‹Ord ν› ν
      by (auto simp: countable_oexp)
    then show ?thesis
      using ‹Ord ν› by (simp add: mult_succ mult.assoc oexp_add)
  qed
qed
```

Suppose (2) holds for some integer $h \geq 1$. Applying the above theorem with $k=2^h$, $\alpha=\omega^{1+\nu h}$, $\beta=\omega^{\nu}$, $\gamma=\omega^{1+\nu}$, we see that (2) also holds with h replaced by $h+1$. Since (2) holds trivially for $h=1$, it follows that (2) holds for all $h<\omega$.

Jean Larson, 1973

$\omega^\omega \longrightarrow (\omega^\omega, m)$ for m a natural number

Proved by CC Chang in a 56-page paper (*J. Combinatorial Theory* A) and generalised by EC Milner

Simplified by Larson to 17 pages, including a new proof of $\omega^2 \longrightarrow (\omega^2, m)$

A few key definitions

Work with *finite increasing sequences*

- $W(n) = \{(a_0, a_1, \ldots, a_{n-1}) : a_0 < a_1 < \cdots < a_{n-1} < \omega\}$ has order type ω^n
- $W = W(0) \cup W(1) \cup W(2) \cup \cdots$ has order type ω^ω

Given $f : [W]^2 \to \{0,1\}$ such that there is no $M \in [W]^m$ s.t. $[M]^2$ is 1-coloured

Show there is a $X \subseteq W$ of order type ω^ω such that $[X]^2$ is 0-coloured

Interaction schemes

For $x, y \in W$, write $x = a_1 * a_2 * \cdots * a_k (*a_{k+1})$ and $y = b_1 * b_2 * \cdots * b_k$

put $c = (|a_1|, |a_1| + |a_2|, \ldots, |a_1| + |a_2| + \cdots + |a_k| (+ |a_{k+1}|))$

define $i(\{x, y\}) = c * a_1 * d * b_1 * a_2 * b_2 * \cdots * a_k * b_k (*a_{k+1})$

(this classifies how consecutive segments in x, y interact)

By Erdős–Milner we can assume $|x| < |y|$

The Nash-Williams partition theorem

A set $A \subseteq W$ is *thin* if for all $s, t \in A$, the sequence s is not an initial segment of t.

Given an infinite set $M \subseteq \omega$, a thin set A, a function $h : \{s \in A : s \subseteq M\} \to \{0,1\}$.

Then there exists an $i \in \{0,1\}$ and an infinite set $N \subseteq M$ so that $h(\{s \in A : s \subseteq N\}) \subseteq \{i\}$.

The three main lemmas

Lemma 3.6. *For every function* $g : [W]^2 \to \{0, 1\}$, *there exists an infinite set* $N \subseteq \omega$ *and a sequence* $\{j_k : k < \omega\}$, *so that for any* $k < \omega$ *with* $k > 0$, *and any pair* $\{x, y\}$ *of form* k *with* $(n_k) < i(\{x, y\}) \subseteq N$, $g(\{x, y\}) = j_k$.

Lemma 3.7. *For every infinite set* N *and every* $m, l < \omega$ *with* $l > 0$, *there is an* m *element set* M, *so that for every* $\{x, y\} \subseteq M$, $\{x, y\}$ *has form* l *and* $i(\{x, y\}) \subseteq N$.

Lemma 3.8. *For any infinite set* $N \subseteq \omega$ *there is a set* $X \subseteq W$ *of type* ω^ω *so that for any pair* $\{x, y\} \subseteq X$, *there is an* $l < \omega$, *so that* $\{x, y\}$ *is of form* l *and if* $l > 0$, *then* $(n_l) < i(\{x, y\}) \subseteq N$.

150 lines, using Nash-Williams

900 lines, including inductive definitions of sequences

1700 lines: more sequences and an order type calculation

… and the main theorem

Now we finish the proof of Theorem 3.1 using these three lemmas. First we apply Lemma 3.6 to f and obtain an infinite set N and a sequence $\{j_k : k < \omega\}$. Then for each $k < \omega$ with $k > 0$, we apply Lemma 3.7 to k, m and $\{n_l : k < l < \omega\}$ and obtain an m element set M_k, so that for any $\{x, y\} \subset M_k$, $f(\{x, y\}) = j_k$. Thus we may conclude that for any $k < \omega$ with $k > 0$, $j_k = 0$. Next we apply Lemma 3.8 to N and obtain a set $X \subseteq W$ of type ω^ω, so that for any $\{x, y\} \subseteq X$, there is an $l < \omega$ for which $\{x, y\}$ has form l and if $l > 0$, then $(n_l) < i(\{x, y\}) \subseteq N$. Thus on pairs $\{x, y\} \subseteq X$ which are not of form 0, $f(\{x, y\}) = j_l = 0$ for some l. By assumption, for any pair $\{x, y\}$ of form 0, $f(\{x, y\}) = 0$, so $f([X]^2) = \{0\}$, and the theorem follows.

150 lines

Why are these machine proofs so long?

- *The level of detail* in published proofs varies **immensely**
- … plus my lack of expertise in the area
- "Obvious" claims—about order types, cardinality, combinatorial intuitions— don't have obvious proofs
- And some of the constructions are gruesome

This sort of inductive definition is tricky!

Let $d^1 = (n_1, n_2, \ldots, n_{k+1}) = (d_1^1, d_2^1, \ldots, d_{k+1}^1)$ and let a_1^1 be the sequence of the first d_1^1 elements of N greater than d_{k+1}^1. Now suppose we have constructed $d^1, a_1^1, \ldots, d^i, a_1^i$. Let $d^{i+1} = (d_1^{i+1}, \ldots, d_{k+1}^{i+1})$ be the first $k+1$ elements of N greater than the last element of a_1^i, and let a_1^{i+1} be the first d_1^{i+1} elements of N greater than d_{k+1}^{i+1}. This defines $d^1, d^2, \ldots, d^m, a_1^1, a_1^2, \ldots, a_1^m$. Let the rest of the sequences be defined in the order that follows, so that for any i and j, a_j^i is the sequence of the least $(d_j^i - d_{j-1}^i)$ elements of N all of which are larger than the largest element of the sequence previously defined:

$$(a_1^m)a_2^1, a_2^2, a_2^3, \ldots, a_2^m, a_3^1, \ldots, a_3^m, \ldots, a_k^1, \ldots, a_k^m, a_{k+1}^m, a_{k+1}^{m-1}, \ldots, a_{k+1}^1.$$

Other formalisations within ALEXANDRIA

- Transcendence of Certain Infinite Series (criteria by Hančl and Rucki)
- Irrationality Criteria for Series by Erdős and Straus
- Irrational Rapidly Convergent Series (a theorem by J. Hančl)
- Counting Complex Roots
- Budan–Fourier Theorem and Counting Real Roots
- Localization of a Commutative Ring
- Projective Geometry
- Quantum Computation and Information
- **Grothendieck Schemes**

Brief remarks on Grothendieck Schemes

- Build-up of mainstream structures in algebraic geometry: presheaves and sheaves of rings, locally ringed spaces, affine schemes
- the *spectrum of a ring* is a locally ringed space, hence an affine scheme
- any *affine scheme* is a scheme
- *They said it couldn't be done* in simple type theory.
- But we did it faster and with less manpower than the Lean guys.
- One key technique: a structuring mechanism known as *locales*.*
- *led by Anthony Bordg*

What can mathematicians expect from proof technology in the future?

- Ever-growing libraries of definitions and theorems
- … with advanced search
- Verification of dull but necessary facts
- … and exhibiting counterexamples
- Detection of analogous developments, with hints for proof steps
- Warnings of simple omissions, e.g. "doesn't S need to be compact?"
- *A careful and increasingly intelligent assistant*